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Auxiliary Problem Concerning Optimal Pursuit on Lagrangian Orbits

Mihai Popescu*
National Institute for Scientific and Technical Creation, Bucharest, Romania

Introduction

A NUMBER of particular cases of pursuit problems have been studied recently. Rozenberg[1] considers a plane pursuit by assuming that the angular velocity of the pursuer is bounded. Simakov[2] determines the rendezvous time when the displacement velocity of the two vehicles and the slopes of the trajectories are subject to constraints. Burrow and Rishel[3] have studied the characteristics of time-optimal trajectories when the acceleration and angular velocity are limited, but the direction of the interception remains free. Reference 4 establishes the conditions necessary for rendezvous in the case of constant velocity when the radius of curvature of the trajectory is subject to a constraint. The studies reported in Refs. 5 and 6 concern the pursuit on Lagrangian orbits of the Earth-moon system assuming that the acceleration is limited.

The present Note tackles time-optimal pursuit by using an auxiliary problem concerning the distance between the two vehicles. The performance index of the auxiliary problem is the value of the distance at the moment when the trajectory reaches the terminal surface S, considered as a convex closed set. We must mention that the derivative of the distance with respect to time does not depend explicitly on the components of the acceleration taken as controls. Thus, the domain of the states where the motion is analyzed is obtained. For the sake of a simple treatment, we take a change of function, defining the difference between the state variables of the pursuer and pursued vehicles.

Formulation of Problem

Let us consider a rotating system of axes with the origin at one of the Lagrangian colinear points of the Earth-moon system. The equations of motion of a space vehicle acted upon by a small propulsion force that moves in orbit around these points have been given in Ref. 7. The motion of two vehicles is governed by the equations

$$\frac{dx_1^j}{dt}=x_2^j, \qquad \frac{dx_2^j}{dt}=k_1x_1^j+2\omega x_4^j+u_1^j$$

$$\frac{dx_3^j}{dt}=x_4^j, \qquad \frac{dx_4^j}{dt}=-2\omega x_2^j+k_2x_3^j+u_2^j \quad (j=1,2) \tag{1}$$

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*Main Research Worker, Aerodynamics Department.

where $X^j(x_1^j,\ldots,x_4^j)$ is a point in the physical space R_4 and $U^j=(u_1^j,\ u_2^j)$ represents the closed domain of the controls, which satisfy the conditions $|u_k^j|\le \bar{u}_k^j$ $(k=1,2)$. The indices 1 and 2 stand for the pursuing and the pursued vehicles, respectively. We note that the components of the velocities and accelerations of the pursuing vehicle are greater than those of the pursued vehicle. Let us put $x_i=x_i^1=x_i^2$ $(i=1,\ldots,4)$. Our aim is to determine the trajectory generated by the controlled system,

$$\frac{dx_1}{dt}=x_2, \qquad \frac{dx_2}{dt}=k_1x_1+2\omega x_4+u_1^1-u_1^2$$

$$\frac{dx_3}{dt}=x_4, \qquad \frac{dx_4}{dt}=-2\omega x_2+k_2x_3+u_2^1-u_2^2 \tag{2}$$

such that the distance between the two vehicles, given by $R_2^2(x)=x_1^2+x_3^2$, is minimal on the terminal surface.

Auxiliary Problem

Assume that the distance between the two vehicles $R_2[x(t)]$ is minimal at $t=t^*$. Then, $\dot{R}_2[x(t^*)]=0$. The system of the equations of motion for the pursuit problem may be written as

$$\frac{dx}{dt}=f(x,u^1,u^2), \quad u^1\in U^1, \quad u^2\in U^2 \tag{3}$$

where x is the four-dimensional vector of the physical space and u^1, u^2 control vectors subject to constraints similar to $u^j\in U^j$. Let us consider the derivative by virtue of Eq. 3 of the expression $\frac{1}{2}R^2[x(t)]$. We have $A=(d/dt)[R_2^2(x)/2]$.

The function $R^2(x,t)$ decreases monotonically with respect to time on each pursuit trajectory. Thus, the domain containing the set of all states possessing this property is $D=\{x;A(x)<0\}$. The terminal surface coincides with the boundary $S=\{x;\dot{R}^2(x)=0\}$ of the domain D. The minimal value of the function $R^2(x)$, denoted by $V(x)$, is attained on the terminal surface S. The pursuer tends to attain the minimal value $V(x)$ and the pursued vehicle tends to maximize this value. Thus, we are led to satisfy the equation

$$\underset{u^1,u^2}{\text{val}}\ [\text{grad}\,V(x),f(x,u^1,u^2)]=0 \tag{4}$$

with the boundary condition on the terminal surface

$$V(x)_{x\in S}=R^2(x) \tag{5}$$

The solution satisfying Eqs. (4) and (5) determines the performance index $V(x)$ only when the trajectory corresponding to this solution permits attaining the terminal surface for every trajectory of the pursued vehicle. The optimal pursuit implies the determination of the time necessary to attain the surface S when the results of the problem of the distance are used. We call this the auxiliary problem.

Optimal Pursuit

Let T be the time necessary for the pursuit trajectory to attain the surface S. The domain of admissible states of the auxiliary problem becomes $D=\{x;x_1x_2+x_3x_4<0\}$. The terminal surface may be expressed in parametric form

$$S=\{x;x_i=s_i\,(i=1,2,3), \qquad s_1s_2+s_3x_4(s)=0\}$$

As the performance index of the auxiliary problem, we take

$$V(x)\,|_s=\tfrac{1}{2}(s_1^2+s_3^2) \tag{6}$$

We consider the case $x_1(S)\neq 0$, $x_3(S)\neq 0$. Expanding Eq. (4) for the auxiliary problem, we obtain

$$\min_{u_k^1}\max_{u_k^2}\left[\frac{\partial V}{\partial x_1}x_2+\frac{\partial V}{\partial x_2}(k_1x_1+2\omega x_4+u_1^1-u_1^2)\right.$$

$$\left.+\frac{\partial V}{\partial x_3}x_4+\frac{\partial V}{\partial x_4}(-2\omega x_2+k_2x_3+u_2^1-u_2^2)\right]=0 \qquad (7)$$

From Eq. (7) one obtains the values of optimal control parameters

$$(u_1^1)^*=-\bar{u}_1^1\ \mathrm{sgn}V_{x2},\qquad (u_2^1)^*=-\bar{u}_2^1\ \mathrm{sgn}V_{x4}$$

$$(u_1^2)^*=\bar{u}_1^2\ \mathrm{sgn}V_{x2},\qquad (u_2^2)^*=\bar{u}_2^2\ \mathrm{sgn}V_{x4}$$

Since

$$\frac{\partial V}{\partial s_j}=\sum_{i=1}^{4}\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial s_j}\qquad (j=1,\ldots,4) \qquad (8)$$

and taking into account the parametric form of S as well as Eqs. (6) and (7), we have

$$[V_{x1}(S),\ldots,V_{x4}(S)]=\mathrm{grad}V(x)/_{x\in S} \qquad (9)$$

By means of the change of variables $\tau=T-t$, the optimal trajectory is obtained by solving the following differential system:

$$\frac{\mathrm{d}x_j}{\mathrm{d}\tau}=-fj[(x_i,(u^1)^*,(u^2)^*]\qquad (i,j=1,\ldots,4) \qquad (10)$$

to which is added the adjoint system

$$\frac{\mathrm{d}V_{xj}}{\mathrm{d}\tau}=\sum_{i=1}^{4}V_{xi}\frac{\partial f_i[(x_j,(u^1)^*(u^2)^*]}{\partial x_j} \qquad (11)$$

Equations (10) and (11) reduce the solution of Eq. (7) to its characteristics.

Determination of Extremals

The characteristic equation attached to the differential system of Eq. (11) is a double-quadratic equation whose roots are $\lambda_{1,2}=\pm a$, $\lambda_{3,4}=\pm ib$, where

$$a=\left[\frac{k_1+k_2-4\omega^2+\sqrt{(k_1+k_2-4\omega^2)^2-4k_1k_2}}{2}\right]^{1/2}$$

$$b=\left[\frac{k_1+k_2-4\omega^2-\sqrt{(k_1+k_2-4\omega^2)^2-4k_1k_2}}{2}\right]^{1/2}$$

Let $\bar{V}_{xi}$ be the particular solution of Eq. (11). By means of the fundamental system of solutions

$$V_{x1}=\bar{V}_{x1},\quad V_{x2}=\bar{V}_{x2}$$

$$V_{x3}=\frac{\bar{V}_{x3}+\bar{V}_{x4}}{2},\qquad V_{x4}=\frac{\bar{V}_{x3}-\bar{V}_{x4}}{2i} \qquad (12)$$

we obtain the general solution of the adjoint system

$$V_{x1}(\tau)=V_1^0a(4\omega^2-k_2+a^2)e^{a\tau}-V_2^0a(4\omega^2-k_2+a^2)e^{-a\tau}$$
$$-V_3^0b(4\omega^2-k_2-b^2)\sin b\tau+V_4^0b(4\omega^2-k_2-b^2)\cos b\tau$$
$$V_{x2}(\tau)=V_1^0(a^2-k_2)e^{a\tau}+V_2^0(a^2-k_2)e^{-a\tau}$$
$$-V_3^0(b^2+k_2)\cos b\tau-V_4^0(b^2+k_2)\sin b\tau$$
$$V_{x3}(\tau)=V_1^0(2\omega k_2)e^{a\tau}+V_2^0(2\omega k_2)e^{-a\tau}$$
$$+V_3^0(2\omega k_2)\cos b\tau+V_4^0(2\omega k_2)\sin b\tau$$
$$V_{x4}(\tau)=V_1^0(2\omega a)e^{a\tau}-V_2^0(2\omega a)e^{-a\tau}$$
$$-V_3^0(2\omega b)\sin b\tau+V_4^0(2\omega b)\cos b\tau \qquad (13)$$

where

$$V_1^0=\frac{2\omega k_2s_1+a(b^2+k_2)s_3}{4\omega k_2a(a^2+b^2)}\qquad V_2^0=\frac{-2\omega k_2s_1+a(b^2+k_2)s_3}{4\omega k_2a(a^2+b^2)}$$

$$V_3^0=\frac{s_3(a^2-k_2)}{2\omega k_2(a^2+b^2)}\qquad V_4^0=\frac{-s_1}{b(a^2+b^2)}$$

The system of Eq. (10) may be represented in vector-matrix form as $x=Ax+b$, where $b=(b_j)$ is the optimal control vector. The solution of the homogeneous system (where $b_j=0$) may be written as

$$\bar{x}_{i(\tau)}=\sum_{k=1}^{4}\bar{C}_k\bar{x}_{ik}(\tau)\qquad (i=1,\ldots,4) \qquad (14)$$

The forms of the particular solutions $\bar{x}_{ik}(\tau)$ are similar to those obtained for V_{xi}, since the characteristic equations for the systems (10) and (11) are identical. We obtain the general solution of the system of Eq. (10) as

$$x_1(\tau)=\bar{C}_1a(4\omega^2-k_2+a^2)e^{a\tau}-\bar{C}_2a(4\omega^2-k_2+a^2)e^{-a\tau}$$
$$+\bar{C}_3b(4\omega^2-k_2-b^2)\sin b\tau+\bar{C}_4b(4\omega^2-k_2-b^2)\cos b\tau$$
$$+\alpha_1(u^1)^*+\beta_1(u^2)^*$$
$$x_2(\tau)=-\bar{C}_1k_1(a^2-k_2)e^{a\tau}-\bar{C}_2k_1(a^2-k_2)e^{-a\tau}$$
$$+\bar{C}_3k_1(b^2+k_2)\cos b\tau+\bar{C}_4k_1(b^2+k_2)\cos b\tau$$
$$+\alpha_2(u^1)^*+\beta_2(u^2)^*$$
$$x_3(\tau)=\bar{C}_1(2\omega k_1)e^{a\tau}+\bar{C}_2(2\omega k_1)e^{-a\tau}$$
$$+\bar{C}_3(2\omega k_1)\cos b\tau+\bar{C}_4(2\omega k_1)\sin b\tau+\alpha_3(u^1)^*+\beta_3(u^2)^*$$
$$x_4(\tau)=-\bar{C}_12\omega k_1ae^{a\tau}+\bar{C}_22\omega k_1ae^{-a\tau}$$
$$+C_32\omega k_1b\sin b\tau-\bar{C}_42\omega k_1b\cos b\tau+\alpha_4(u^1)^*+\beta_4(u^2)^* \qquad (15)$$

where α_i and β_i $(i=1,\ldots,4)$ are constants whose expressions are given in the Appendix. The values of the optimal control for Eq. (10) are

$$(u^1)^*=-(\bar{u}_1^1+\bar{u}_1^2)\ \mathrm{sgn}V_{x2}$$
$$(u^2)^*=-(\bar{u}_2^1+\bar{u}_2^2)\ \mathrm{sgn}V_{x4} \qquad (16)$$

Taking into account Eq. (9), from the adjoint system of Eq. (11) it follows that

$$\left.\frac{\mathrm{d}V_{x2}}{\mathrm{d}\tau}\right|_S=s_1,\qquad \left.\frac{\mathrm{d}V_{x4}}{\mathrm{d}\tau}\right|_s=s_3$$

The determination of the solution of Eqs. (10) and (11) with initial conditions on s allows us to establish the form of the optimal pursuit trajectory in the neighborhood of s. On this trajectory, the controls do not alter their sign and they take the sign of $\dot{V}_{x2}$ and $\dot{V}_{x4}$ on s. The conditions on the terminal surface become initial conditions for Eq. (10), such that we have $x_i(0)=s_i$ and $x_4(0)=-s_1s_2/s_3$ (i = 1,2,3).

Taking into account Eq. (15), one determines the integration constants $\bar{C}_i\,(i=1,\ldots,4)$ as

$$\bar{C}_1 = \frac{2\omega k_1 s_1 s_3 - 2\omega a s_2 s_3 + a(b^2+k_2)s_3^2 - (4\omega^2 - k_2 - b^2)s_1 s_2}{4\omega k_1 a(a^2+b^2)s_3}$$

$$\bar{C}_2 = \frac{-2\omega k_1 s_1 s_3 - 2\omega a s_2 s_3 + a(b^2+k_2)s_3^2 + (4\omega^2 - k_2 - b^2)s_1 s_2}{2\omega k_1 a(a^2+b^2)s_3}$$

$$\bar{C}_3 = \frac{2\omega s_2 + (a^2-k_2)s_3}{2\omega k_1(a^2+b^2)}$$

$$\bar{C}_4 = -\frac{2\omega k_1 s_1 s_3 - (4\omega^2 - k_2 + a^2)s_1 s_2}{2\omega k_1 b(a^2+b^2)s_3} \tag{17}$$

With the notation $\gamma_i = \alpha_i(u^1)^* + \beta_i(u^2)^*$, Eq. (15) becomes

$$\Sigma \bar{C}_k x_{ik}(\tau) = x_i - \gamma_i \tag{18}$$

with the unknowns $e^{a\tau}$, $e^{-a\tau}$, $\sin b\tau$, and $\cos b\tau$. Performing the calculations, we obtain the analytical expression of the pursuit time on the optimal trajectory

$$\begin{aligned}\tau(x) = {} & (1/d)\ell n\{[2\omega k_1(x_1-\gamma_1) + 2\omega a(x_2-\gamma_2) \\ & + a(b^2+k_2)(x_3-\gamma_3) + b(4\omega^2-k_2-b^2)(x_4-\gamma_4)]s_3\} \\ & \div [2\omega k_1 s_1 s_3 - 2\omega a s_2 s_3 + a(b^2+k^2)s_3^2 \\ & - (4\omega^2 - k_2 - b^2)s_1 s_2]\end{aligned} \tag{19}$$

An analysis of the motion for the entire time devoted to the pursuit is made by taking into account the expressions for the control functions. For the analyzed motion, the values of the physical coordinates x_i $(i=1,\ldots,4)$ have to fulfill condition $x_1 x_2 + x_3 x_4 < 0$. After the determination of the domain D, the set of the admissible states for the pursuit problem coincides with the domain of the states for the auxiliary problem. A particular case of the analyzed problem is obtained when the two vehicles meet on the terminal surface for $s_1 = s_3 = 0$. The domain of the admissible states for this case is analogous to the one above.

Appendix

Expressions for the terms of Eq. (15) are as follows:

$$\alpha_1 = \frac{b^2(4\omega^2-k_2-b^2) - a^2(a^2+b^2)}{k_1[4\omega^2(k_2-a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\beta_1 = -\frac{(a^2+b^2)(4\omega^2/b - k_2 + a^2)(4\omega^2 - k_2 - b^2 - 1)}{4\omega a k_1[4\omega^2(k_2-a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\alpha_2 = \frac{b^2(a^2-k_2)}{a[4\omega^2(k_2-a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\begin{aligned}\beta_2 = {} & \{(a^2+b^2)[2a^2(b^2+k_2)(4\omega^2-k_2+a^2) \\ & - b^2(a^2-k_2)(4\omega^2-k_2-b^2+1)\} \\ & \div 4\omega a^2 b^2[4\omega^2(k_2-a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]\end{aligned}$$

$$\alpha_3 = \frac{2\omega a}{[4\omega^2(k_2-a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\beta_3 = \frac{(a^2+b^2)[b^2(4\omega^2-k_2-b^2+1) + 2a^2(4\omega^2-k_2+a^2)]}{2\omega a^2 b^2[4\omega^2(k_2-a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\alpha_4 = -\frac{2\omega b^2}{[4\omega^2(k_2-a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

$$\beta_4 = \frac{(a^2+b^2)(4\omega^2-k_2-b^2-1)}{2a[4\omega^2(k_2-a^2) + 2a^2(b^2+k_2) + b^4 - k_2^2]}$$

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Comparison of Local Pole Assignment Methods

P.T. Kabamba* and K.C. Shin†

University of Michigan, Ann Arbor, Michigan

I. Introduction

CONSIDER the following standard time-invariant, minimal linear dynamical system

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector; $y\,\mathbb{R}^r$ is the output vector; A, B, and C are matrices of appropriate dimensions. System (1,2) could be controlled by state feedback of the form $u = Fx$, but such a control would be unfeasible if the state vector is not accessible. This difficulty can be circumvented by including an estimator in the controller and feeding back the state vector estimate, but this would require a possibly cumbersome controller. It has been suggested to control system (1,2) by direct output feedback of the form $u = Py$, in which case the closed loop satisfies

$$\dot{x} = (A + BPC)x \tag{3}$$

The dynamical behavior of Eq. (3) will depend on the location of the eigenvalues of $A + BPC$. The problem of pole assignment by direct-output feedback has received substantial

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*Department of Aerospace Engineering.

†Department of Mechanical Engineering.